

# **Real-World Quantitative Finance**

(A Poor Man's Guide To What Physicists Do On Wall St.)

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## **Introduction**

- Models in Physics
- Models in Finance
- Two Preambles
- Ten Practical Principles of Financial Modeling

## Models in Physics

- **Fundamental Models or Theories**

Fundamental models attribute effects to deep dynamical causes.

Kepler's laws of planetary motion – not quite a theory:

- planets move about the sun in elliptical orbits;
- the line from the sun to the planet sweeps out equal areas in equal times; and
- $(\text{period})^2 \sim (\text{radius})^3$ .

Nevertheless, the laws do provide profound insight.

Newton adds dynamics, provides a fundamental theory.

- **Phenomenological Models**

A toy or analogy to help visualize something unobservable.

“As-if” models.

- Liquid drop model of nucleus;

Calibrate to known phenomena, then use it to predict the unknown.

## Models in Finance

There is no fundamental theory in finance. There are no laws.  
That's why the textbooks are so mathematically rigorous.

- **Phenomenological Models**

Models in finance are used to *turn opinions into prices*, and sometimes to turn prices into implied opinions.

Most financial models determine the value of a security by assuming that investors make *simple rational assessments*, based on a few *intuitively understandable variables* which represent the market's "*opinion*" of the *future*: dividends, interest rates, volatilities, correlations, default rates, etc. Models are causal and perturbative.

The *implied values* of these variables are determined by calibrating/renormalizing the model to liquid market prices.

Prices are often non-linear in these variables, while intuition about their value is more reliably linear. One uses the models to interpolate smoothly from known to unknown.

- **Statistical Models**

Not models in the physics sense.

Physicists use statistics to test theories.

Economists use it to find relationships and so make "theories."

Mostly regression without explicit dynamics, and therefore non-perturbative.  
Useful when you *have* to have some estimate.

## Two Preambles

- Preamble 1.

It's common to imagine that physicists on Wall St. spend their time predicting the future. They rarely do.

Most of us working in quantitative *haute couture*, using models to create custom-tailored risk profiles from off-the-shelf products.

- Preamble 2.

Pure arbitrage is simultaneously buying at one price and selling at another. It's amazingly rare.

What's more sloppily called arbitrage is finding a discrepancy between a model price and a market price, and acting on it.

## 1. Use rates of growth to quote the values of riskless securities.

The present value of a certain payment of \$1 at future time  $t$

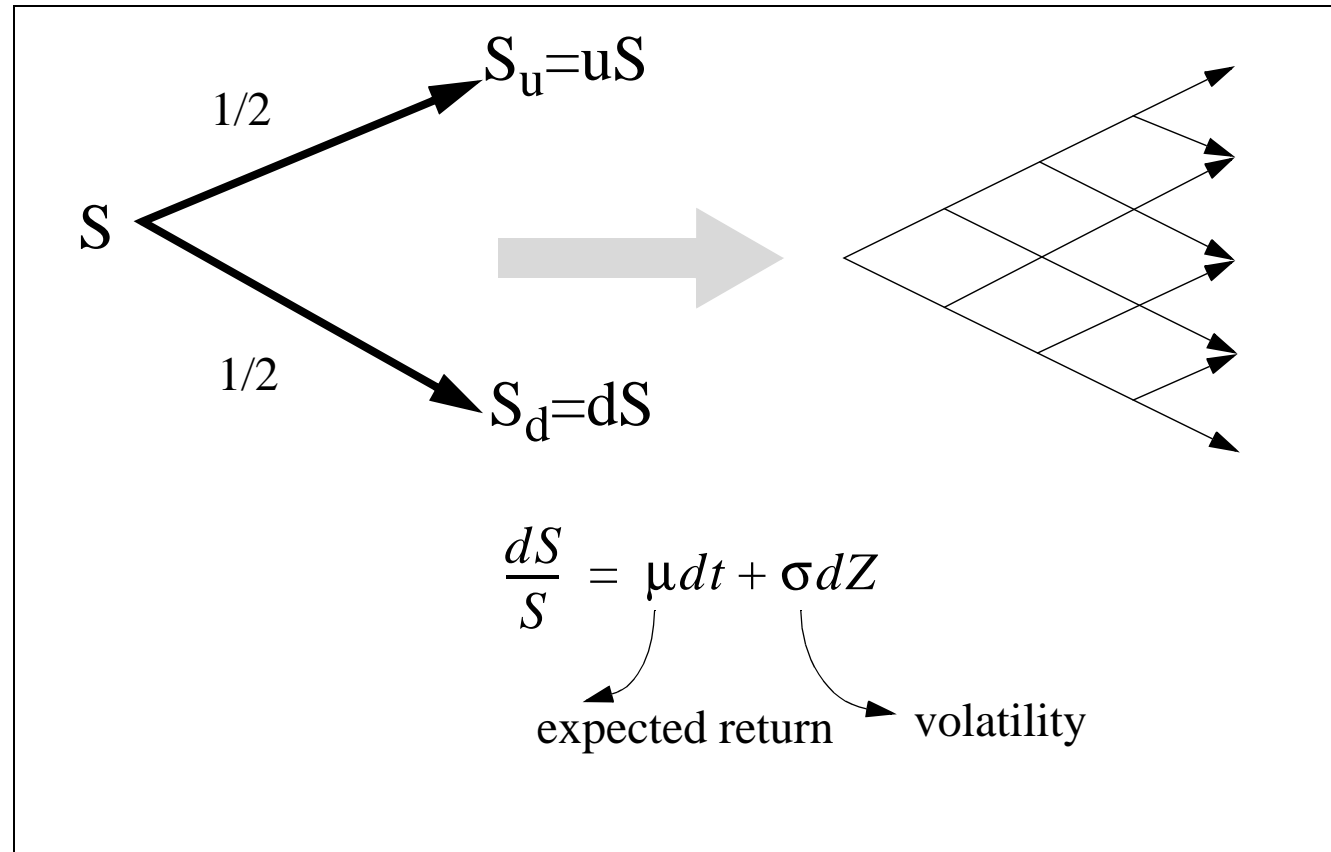
$$P(t) = \exp[-r_t t] \quad \text{discount factor}$$

**Rates of growth**  $r$  is the right conceptual variables, just as velocity is the right variable to look at if you want to know how long a car will take to get somewhere.

**Forward rates** are the future rates of growth  $f$  you can lock in by buying and selling today. A very important way of thinking.

## 2. Model risky securities via uncertain growth.

Simple model: assume probability of an up or down move is 1/2.



One *can* build more sophisticated models of uncertainty.

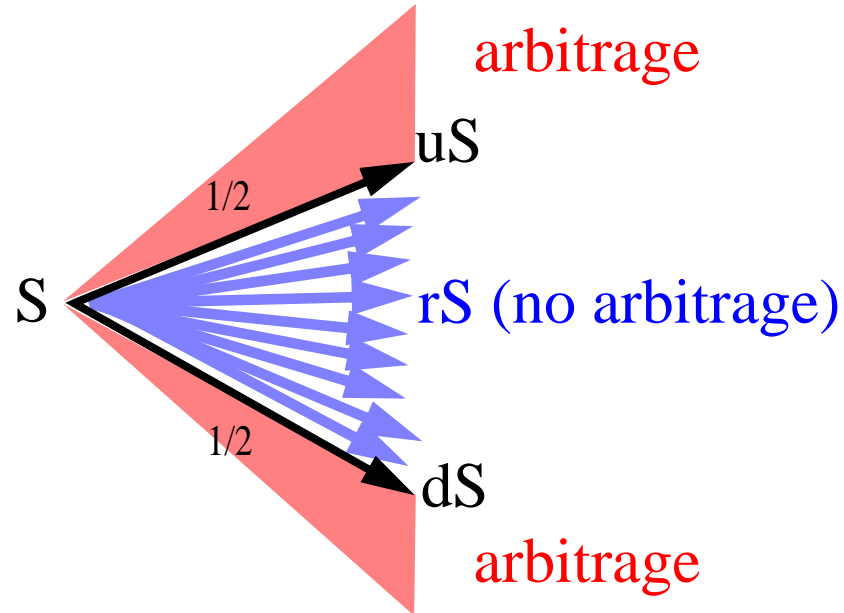
- Stochastic calculus:

$dZ$  is a Brownian motion: mean zero, standard deviation  $\sqrt{t}$

$$dS \sim dt \quad dS^2 \sim dt$$

### 3. Avoid Riskless Arbitrage

A risky portfolio *must* bear risk.



Its potential growth rates must bracketed the riskless rate.

Therefore, the riskless return  $r$  is a convex combination of  $u$  and  $d$ .

$$pu + (1 - p)d = r$$

$p \equiv (1/2 + \lambda)$  is a probability measure *for each stock*.

This probabilistic thread runs throughout modern finance.

$\lambda$  is the *risk premium* of the stock.

Remember: “No arbitrage” is a constraint on *how we model the world*.

## 4. The Fundamental Question: What Expected Growth $\mu$ Should You Expect for a Given Risk $\sigma$ ?

- Use The Invariance Principle: *Two portfolios with the same perceived instantaneous risk should have the same expected return.*

A particular low-risk stock can be duplicated by a weighted portfolio of a high-risk stock and a riskless bond. Both must then have the same risk and return.

This can only hold if

$$\frac{\mu - r}{\sigma} = \lambda$$

Excess return per unit of risk is the same for all stocks, and is equal to the risk premium  $\lambda$ .

- More risk, more return.
- What is the value of  $\lambda$ ?

## 5. Diversification

- The removal of risk by the law of large numbers.

If

you can buy a very large portfolio of stocks,  
and their returns are uncorrelated

Then

asymptotically the portfolio volatility  $\sigma \rightarrow 0$   
and so its return  $\mu \rightarrow 0$ ,

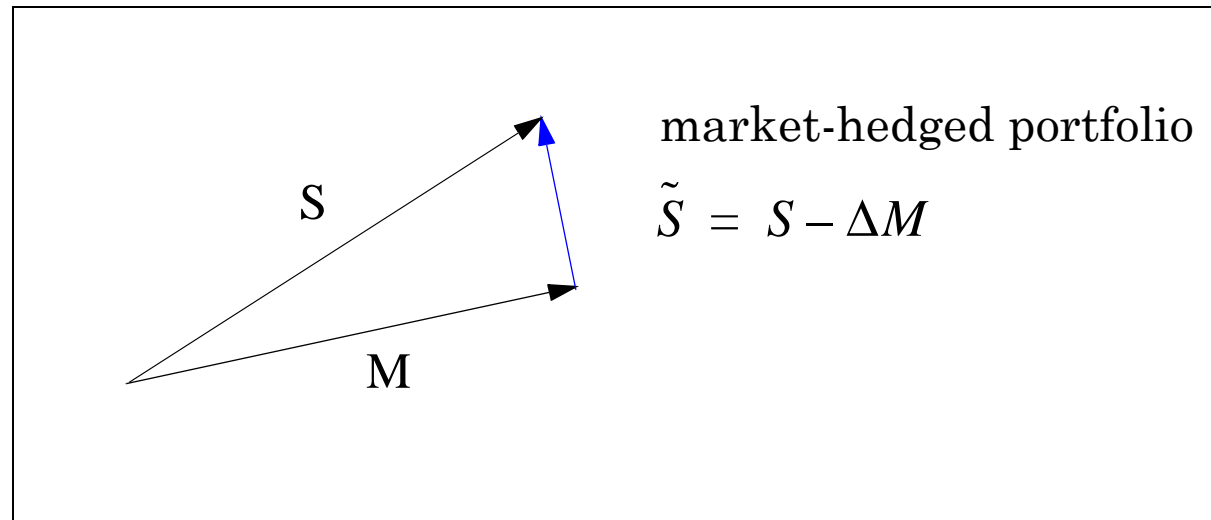
Therefore, for each uncorrelated security

$\lambda = 0$       Zero risk premium, expect riskless growth

- You are not paid to take on diversifiable risk.

## 6. Hedging

- The removal of risk by cancellation of common factors.
- Example: all stocks are correlated with the “market”  $M$ . You can remove this component of risk from any stock by combining it with a short position in the market.
- All that’s left in the hedged portfolio  $\tilde{S}$  is the residual, market-orthogonal risk:



## 7. Capital Asset Pricing Model

- For stocks correlated with the overall market:.

If

you remove the overall market risk from each stock

Then

same residual risk must lead to the same expected return.

Diversifying over the uncorrelated residual risk of a large portfolio, we obtain

$$(\mu_S - r) = \beta_{SM}(\mu_M - r)$$

$$\beta_{sm} = \sigma_{SM} / \sigma_M^2$$

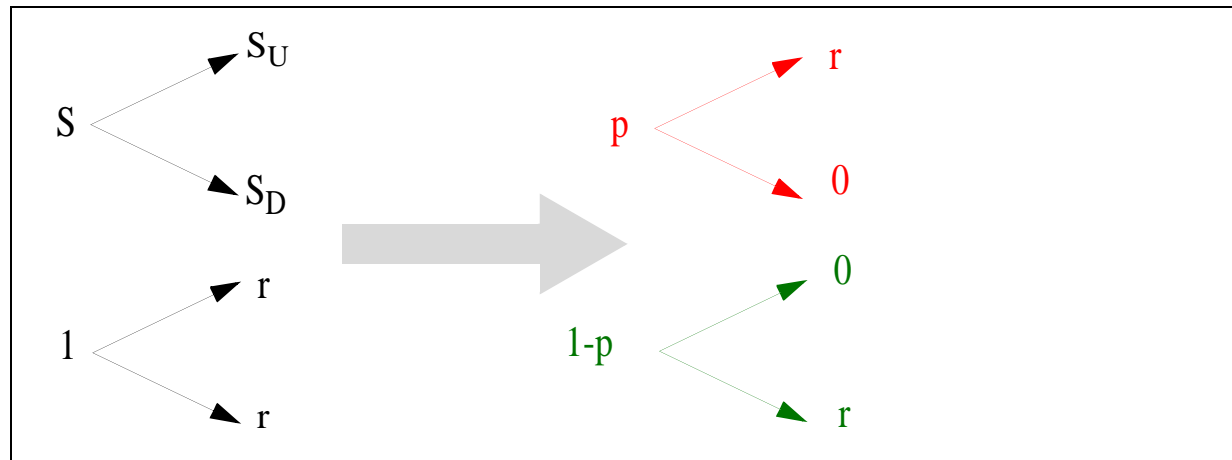
The return of a stock is proportional to the market's return times its comovement with the market.

- You are not paid to take on diversifiable risk.

## 8. Options are not independent securities

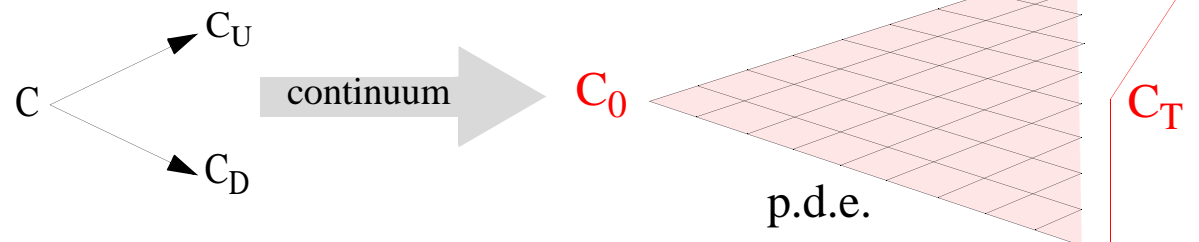
Options (derivatives) have pay-offs which are non-linear functions of the underlying stock price.

A stock and a bond can be decomposed into Arrow-Debreu securities  $p$  and  $1-p$  that span the price space for a short time  $\Delta t$ :



With these one-state securities  $p$  and  $1-p$ , you can dynamically replicate the payoff of the non-linear option  $C(S)$  at each instant.

$$rC = p C_U + (1-p) C_D$$



Since you can replicate, you don't care about path of stock, only its volatility.  
Options traders bet on (trade) volatility.

## 9. Extensions of replication (the past 25 years)

- Extension of the Black-Scholes-Merton replication method of pricing derivatives on currencies, commodities, **interest-rate-sensitive securities**, mortgages, credit-derivatives, etc.

### Strategy:

- build a realistic model of the stochastic behavior of the underlying security;
  - Calibrate it to the current prices of the underlying security;
  - Figure out how to replicate the derivative security;
  - Find the value of the derivative by backward induction or Monte Carlo simulation of the replication process.
- Traders have become more analytical as they realize that they are trading volatility, and develop both an understanding and a feel for the model.
  - Markets are able to *estimate* the value of exotic and hybrid options.
  - Perturbation of the idealized Black-Scholes model to take account of the real world and *behavioral and perceptual issues*:
    - illiquidity, transactions cost, noncontinuous trading, skew, non-normal distributions, market participants' behavior, etc.

## 10. One last trick: change of numeraire

- The most ubiquitous trick in finance.

The value of an instrument should be independent of the currency you choose to model it in.

The currency can be a dollar, a yen, the value of an IBM share, the value of anything tradeable.

Choosing an currency or numeraire allows you to make simpler models for complex products.

It's a bit like choosing a gauge in field theory, but less complicated.

## How do you tell when a model is right?

- Finance isn't physics.
- Value is determined by human behavior, which is not time-invariant.
- Models as gedanken experiments, theoretical laboratories or parallel thought universes for testing cause and effect. Models provide a common language to communicate opinions and values. No single one is right.
- Fischer Black (1986):

In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant.